Chapter 4

Numerical session on matlab

Chapter 4

- Introduction (p. 3)
- Simple Gibbs and MH MCMC (p. 9)
- CP-AR models (p. 27)
  - Carlin, Gelman and Smith : Griddy-Gibbs
  - Chib's algorithm

Introduction

Introduction

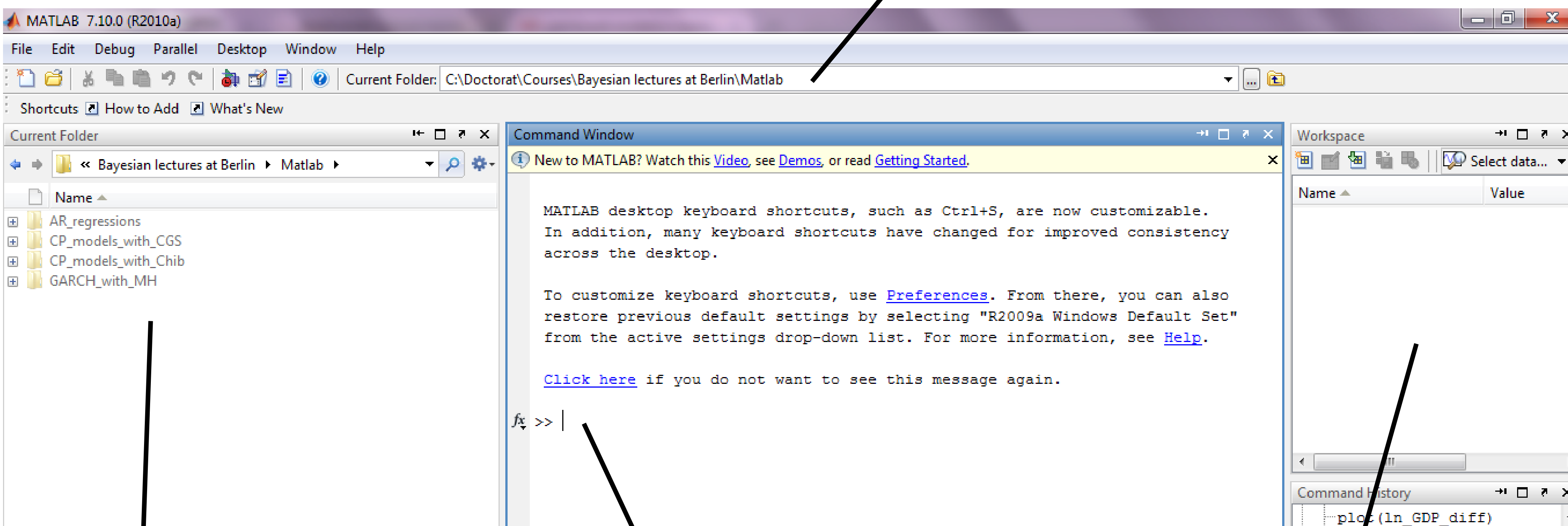
Quick start

- Four directories :
  - AR_regressions – ***MCMC for simulating posteriors of AR models***
  - GARCH_with_MH – ***MH MCMC for simulating posteriors of a GARCH(1,1) model***
  - CP_models_with_CGS – ***MCMC for drawing posteriors of AR models with 2 regimes using the Carlin, Gelman and Smith approach***
  - CP_models_with_Chib – ***MCMC for posterior distributions of AR models with k regimes based on Chib's algorithm***

Introduction

How to use the programs ?

Current folder : Matlab directory



Navigator

Command window

Workspace

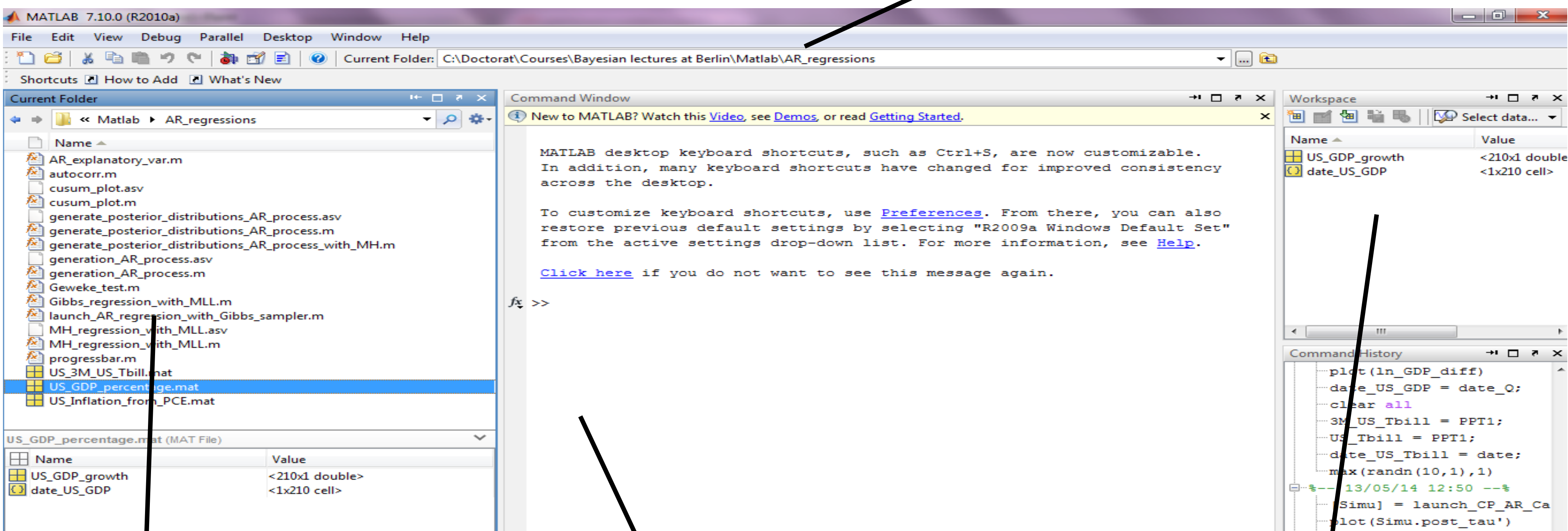
Introduction

Go to the AR_regressions directory

Click on US_GDP_percentage

Current folder :

Matlab directory



Navigator

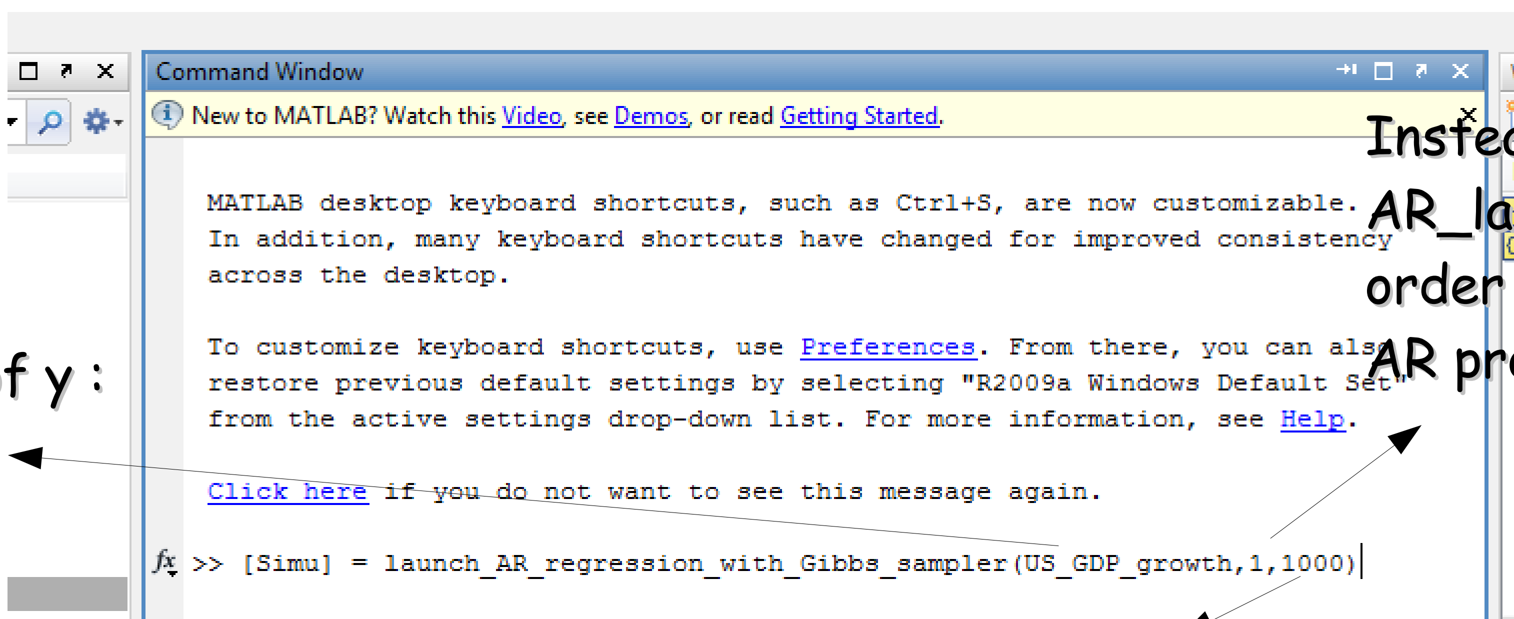
Command window

Workspace with data

Introduction

How to run a program ?

- Open the program : launch_AR_regression_with_Gibbs_sampler
  - Comments to run the function are in green
- Copy and paste the first line without 'function' in the command window



Instead of y : the data

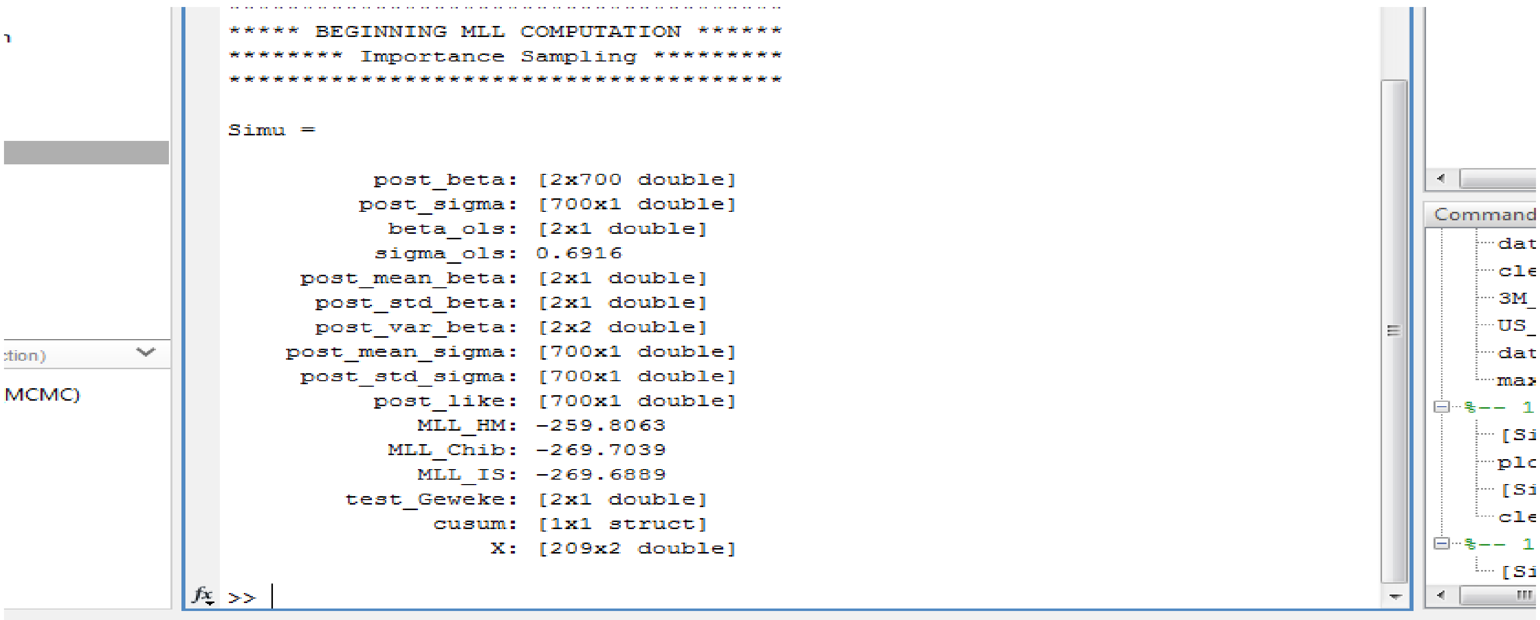
Instead of AR_lags : the order of the AR process

nb_MCMC : 1000 iterations

Introduction

What is a structure ?

- Simu is a matlab structure containing many tables



- To access the matrix 'post_beta' :

→ Type in command window : Simu.post_beta

Simple MCMC

Gibbs sampler

- **The model**

$$\begin{cases} y_t &= \beta' x_t + \epsilon_t \\ \epsilon_t &\sim \text{i.i.d.} N(0, \sigma^2) \end{cases}$$

- **The prior distributions**

$$\beta \sim N(\beta_0, \Sigma_0) \qquad \sigma^2 \sim IG(IG_a, IG_b)$$

- **Conditional posterior distributions :**

$$\pi(\theta | Y_{1:T}, \sigma^2) \sim N(\bar{\mu}, \bar{\Sigma}) \qquad \begin{cases} \bar{\Sigma} = [\sigma^{-2} \sum_{t=1}^{T} (x_t x_t') + \Sigma_0^{-1}]^{-1} \\ \bar{\mu} = \bar{\Sigma}[\sigma^{-2} \sum_{t=1}^{T} x_t y_t + \Sigma_0^{-1} \beta_0] \end{cases}$$

$$\pi(\sigma^2 | Y_{1:T}, \theta) \sim IG(IG_a + T/2, IG_b + \sum_{t=1}^{T} \epsilon_t^2 / 2)$$

Gibbs sampler

- **Initial values :**

MLE estimates

$$\beta = \left(\sum_{t=1}^{T} x_t x_t'\right)^{-1} \left(\sum_{t=1}^{T} x_t y_t\right)$$

$$\sigma^2 = \sum_{t=1}^{T} (y_t - \beta' x_t)^2 / T$$

- **Gibbs sampler :**

$$\beta \sim \beta | Y_{1:T}, \sigma^2 \qquad \sigma^2 \sim \sigma^2 | Y_{1:T}, \beta$$

Discard the first draws as burn-in period

Use the rest as a sample of the posterior distribution :

$$\beta, \sigma^2 | Y_{1:T}$$

Gibbs sampler

- **Open the function :** ***Gibbs_regression_with_MLL***
  - Gibbs sampler of a simple regression
  - Provide the marginal log-likelihood (MLL) estimated from three different approaches (Chib, Importance sampling and Harmonic mean)

→ **Harmonic mean : very bad estimate**

- **The prior distributions**

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%% Set the prior values
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%% Prior :
%%% beta ~ N(beta_0,Sigma_0)
%%% sigma^2 ~ IG(a,b)
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
IG_b = 1;
IG_a = 1;
var_uninformative = 100; %% We fix the variance of each beta equal to
beta_0 = zeros(dimension,1);
Sigma_0 = diag(ones(dimension,1)*(var_uninformative));
inv_Sigma_0 = diag(ones(dimension,1)*(1/var_uninformative));
```

Hyper-parameters of the mean parameters

Hyper-parameters of the variance

Gibbs sampler

```
%%%%%%%%%%%%%%%%
%%%% MCMC starting values
%%%%%%%%%%%%%%%%
beta = beta_ols;
sigma = sigma_ols;
iter = 1;
for i=1:nb_MCMC
    progressbar(i/nb_MCMC);
    %%%%%%%%%%%%
    %%%% Saving posterior samples
    %%%%%%%%%%%%
    if(i>burn_in)
        beta_post(:,iter) = beta;
        sigma_post(iter) = sigma;
        like_post(iter) = (-T/2)*log(2*pi*sigma) - (y-X*beta)'*(y-X*beta)/(2*sigma);
        iter = iter +1;
    end
    %%%%%%%%%%%%%%
    %%% Update of sigma|beta
    %%%%%%%%%%%%%%
    IG_post_b_MCMC = IG_b + 0.5*sum((y-X*beta).^2);
    inv_sigma = gamrnd(IG_post_a_MCMC,1/IG_post_b_MCMC);
    sigma = 1/inv_sigma;
    %%%%%%%%%%%%%%
    %%% Update of beta|Sigma
    %%%%%%%%%%%%%%
    Sigma_cond = inv(inv_sigma*Mat_XX + inv_Sigma_0);
    mu_cond = Sigma_cond*(inv_sigma*Vec_Xy + inv_Sigma_0*beta_0);
    beta = mvnrnd(mu_cond,Sigma_cond)';
end
%%%%%%%%%%%%%%
```

Starting values

Saving posterior draws

First block - the variance

First block - the mean parameters

Gibbs sampler

```
%%%%%%%%%%%%%%%
%%%% MCMC starting values
%%%%%%%%%%%%%%%
beta = beta_ols;
sigma = sigma_ols;
iter = 1;
for i=1:nb_MCMC
    progressbar(i/nb_MCMC);
    %%%%%%%%%%%
    %%%% Saving posterior samples
    %%%%%%%%%%%
    if(i>burn_in)
        beta_post(:,iter) = beta;
        sigma_post(iter) = sigma;
        like_post(iter) = (-T/2)*log(2*pi*sigma) - (y-X*beta)'*(y-X*beta)/(2*sigma);
        iter = iter +1;
    end
    %%%%%%%%%%%%%
    %%% Update of sigma|beta
    %%%%%%%%%%%%%
    IG_post_b_MCMC = IG_b + 0.5*sum((y-X*beta).^2);
    inv_sigma = gamrnd(IG_post_a_MCMC,1/IG_post_b_MCMC);
    sigma = 1/inv_sigma;
    %%%%%%%%%%%%%
    %%% Update of beta|Sigma
    %%%%%%%%%%%%%
    Sigma_cond = inv(inv_sigma*Mat_XX + inv_Sigma_0);
    mu_cond = Sigma_cond*(inv_sigma*Vec_Xy + inv_Sigma_0*beta_0);
    beta = mvnrnd(mu_cond,Sigma_cond)';
end
%%%%%%%%%%%%%
```

Starting values

Saving posterior draws

First block - the variance

First block - the mean parameters

Gibbs sampler

- **Open the function :** ***launch_AR_regression_with_Gibbs_sampler***

In the command window, type :

[Simu] = launch_AR_regression_with_Gibbs_sampler(US_GDP_growth,1,3000,1)

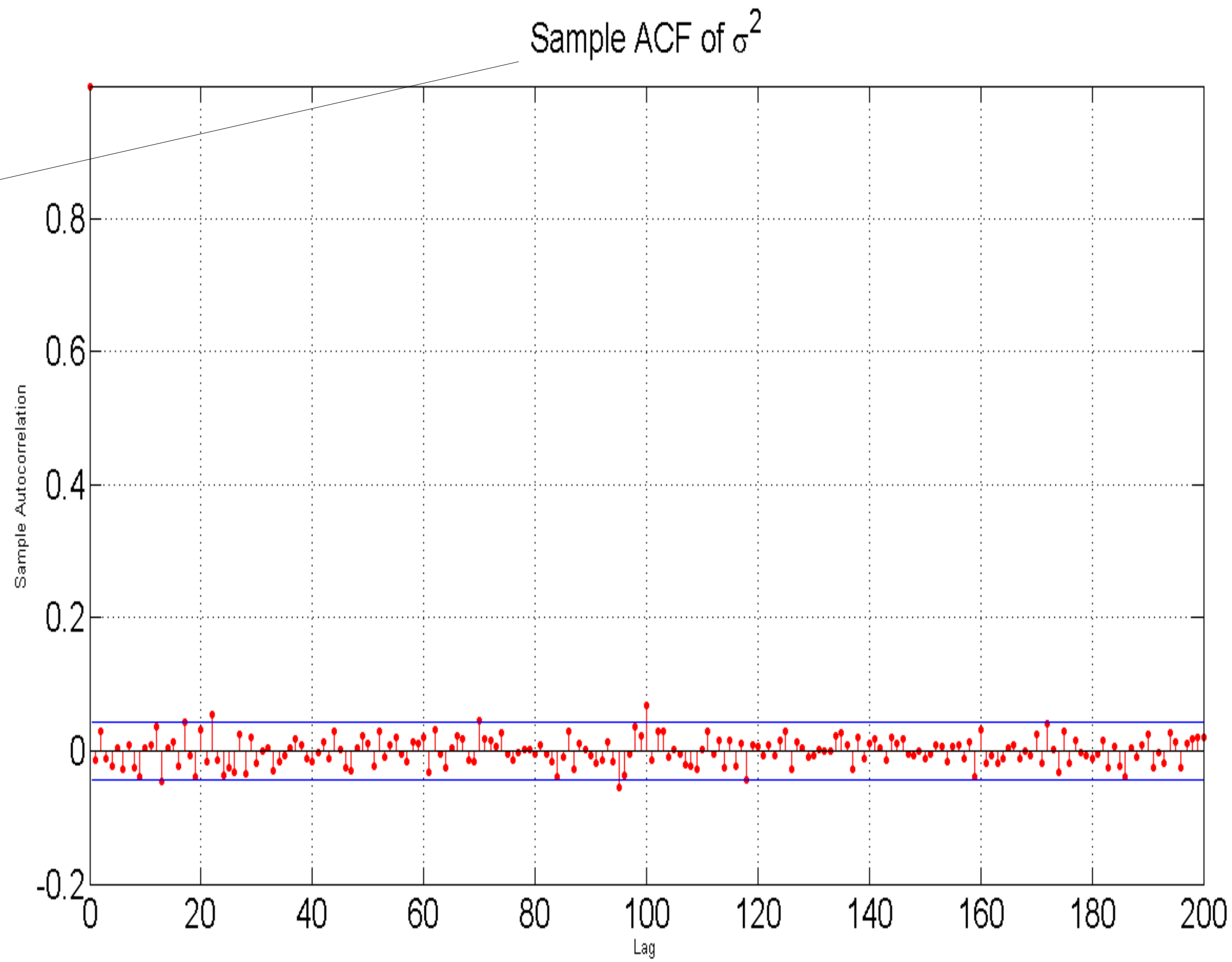
- The data : US_GDP_growth
- AR order : 1
- Nb MCMC iterations : 3000
- Convergence Graphics : on

Gibbs sampler

- **Graphics :**

  Empirical correlations between MCMC draws

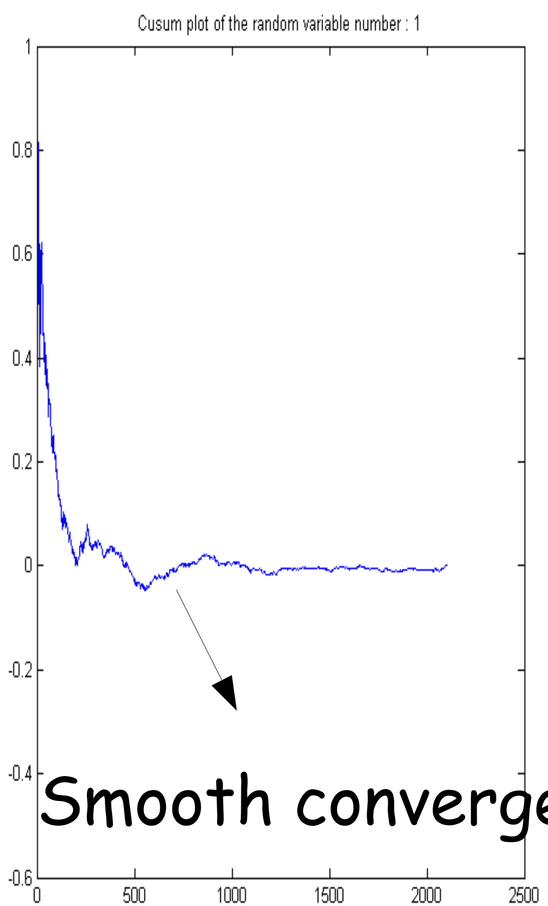
Variance

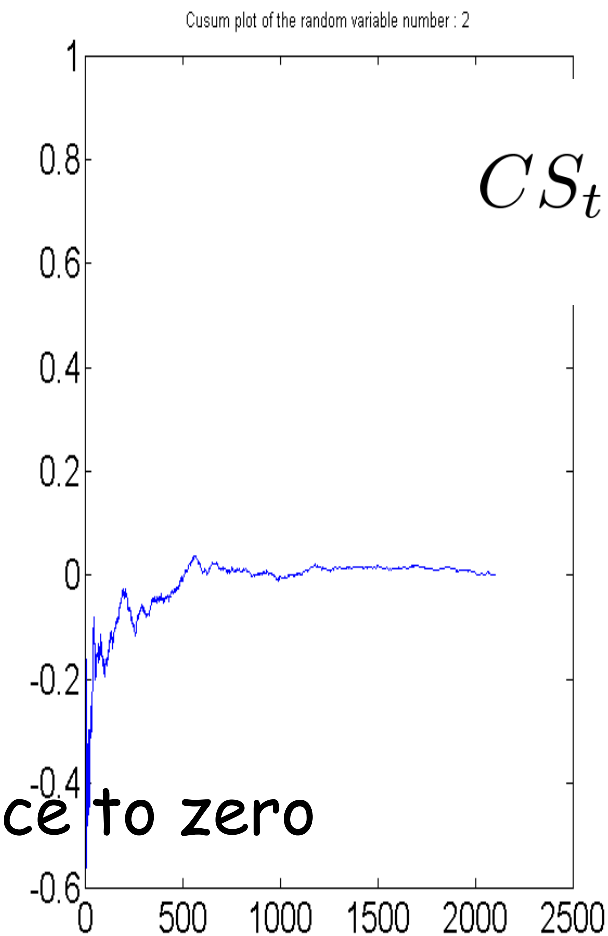


Gibbs sampler

- **Graphics :**

  Cumsum plots of the two mean parameters





Smooth convergence to zero

Empirical std of the parameter

$$CS_t = \left(\frac{1}{i}\sum_{j=1}^{i}\theta^j - m_\theta\right)/s_\theta$$

Empirical mean of the parameter

$$CS_N = 0$$

Gibbs sampler

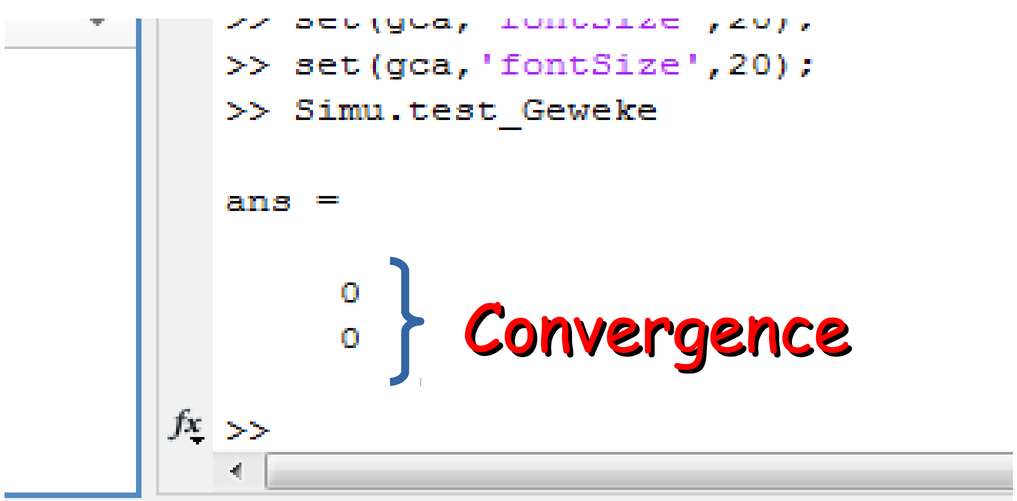
- **Geweke's test of MCMC convergence :**

Comparison of two empirical means well separated

In the command window :

Simu.test_Geweke

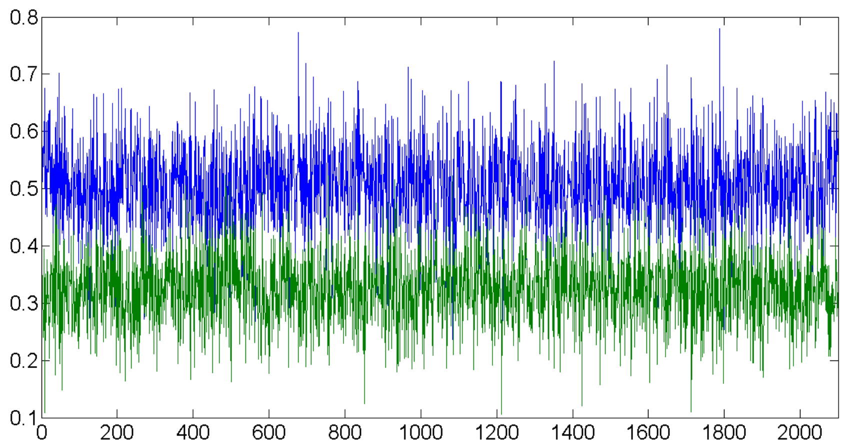
→ Test for each mean parameters and return zero if the hypothesis is not rejected at 95%



Gibbs sampler

- **Results :**

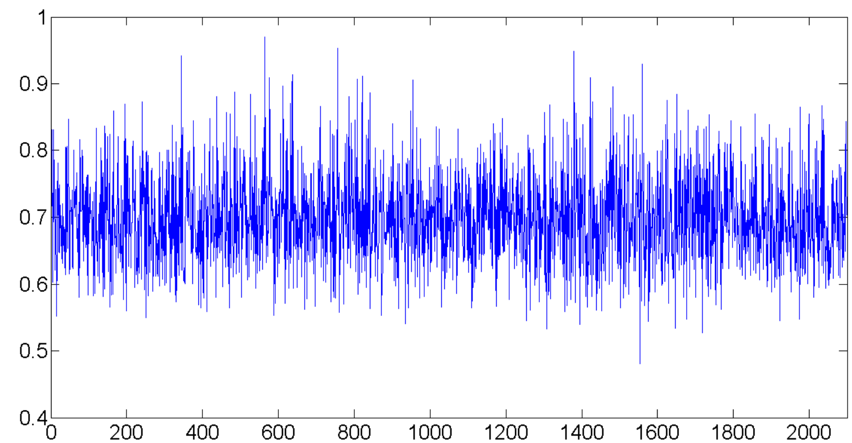
plot(Simu.post_beta')



mean(Simu.post_beta')= 0,50 ; 0,32

std(Simu.post_beta') = 0,08 ; 0,06

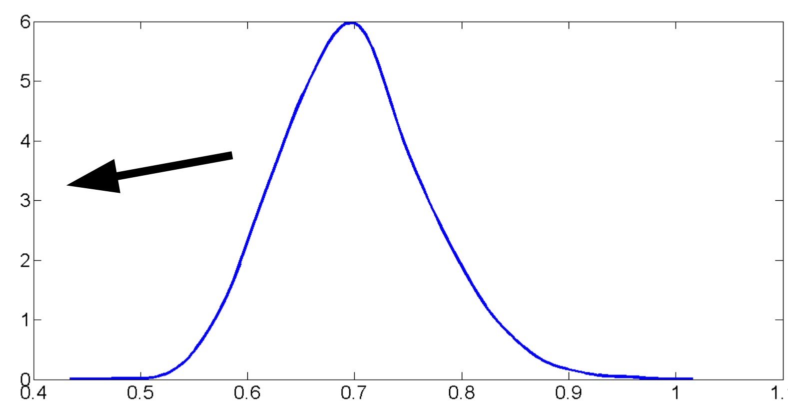
plot(Simu.post_sigma')



mean(Simu.post_sigma')= 0,70

std(Simu.post_sigma') = 0,07

Marginal distribution of the variance



Gibbs sampler

- Testing for the order of the AR process :
  - Launch several times the program with different # of lags
  - Compare the MLL obtained from the different simulations

	AR(0)	AR(1)	AR(2)	AR(3)	AR(4)
MLL HM	-274,16	-259,89	-256,36	-257,16	-254,75
MLL Chib	-279,12	-269,7	-269,71	-273,72	-275,46
MLL IS	-279,12	-269,7	-269,71	-273,72	-275,46

- Harmonic mean : not reliable
- Same Estimates from the local and the global formula

MH sampler

- **Open the function :** ***MH_regression_with_MLL***
  - MH sampler of a simple regression
  - Provide the marginal log-likelihood (MLL) estimated from three different approaches (Chib, Importance sampling and Harmonic mean)

- **Running an estimation of AR model with MH sampler :** ***launch_AR_regression_with_Gibbs_sampler***

Gibbs sampler

- **Adaptation of the proposal distribution**

(Atchadé and Rosenthal (2005))

$$\bar{\Sigma}_i = \bar{\Sigma}_{i-1} + (\phi_r^{i-1} - \phi_{target})/(i^c) \;\; \text{if} \;\; \Sigma_{Low} < \bar{\Sigma}_i < \Sigma_{High}$$

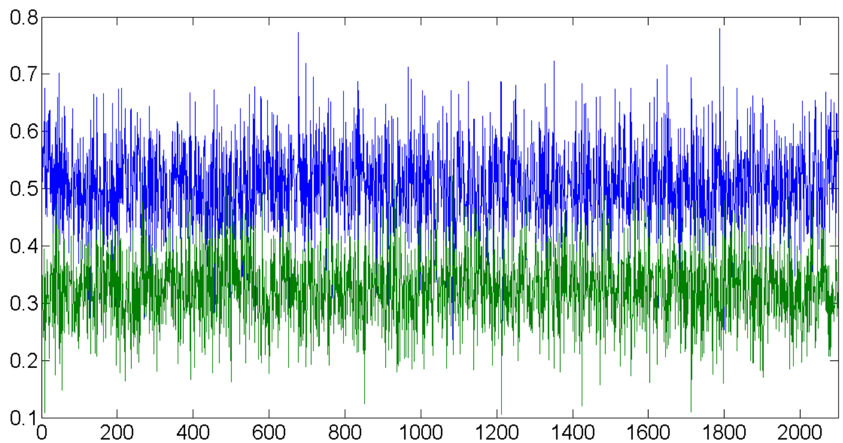
```
for q=1:dimension+1
    adapt_rate(q) = max(min_max_var(1),adapt_rate(q) + (accept_rate(q)/i - 0.45)/(i^eta));
    if(adapt_rate(q)>min_max_var(2))
        adapt_rate(q) = min_max_var(2);
    end
end
```

ϕ_{target}

Comparison Samplers

- AR(1)

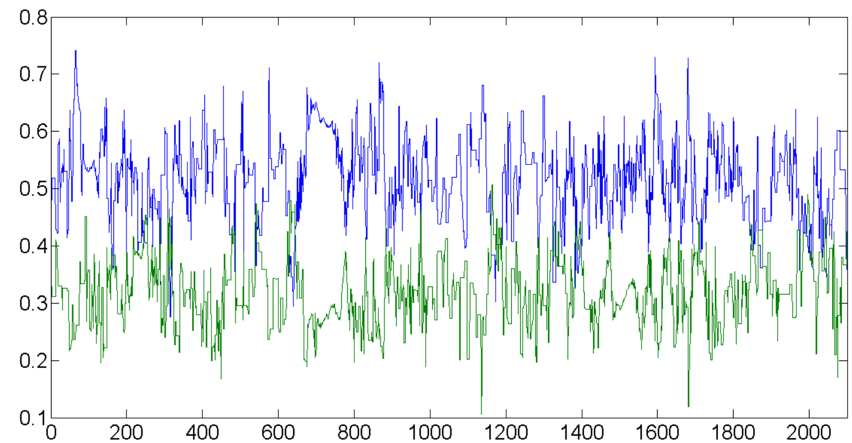
Gibbs sampler



Mean= 0,50 ; 0,32

Std = 0,08 ; 0,06

Metropolis-Hastings



Mean = 0,51 ; 0,32

Std = 0,07 ; 0,06

Acceptance rate :

45 % ; 44 % ; 45 %

MH sampler

- Testing for the order of the AR process :
  - Launch several times the program with different # of lags
  - Compare the MLL obtained from the different simulations

	AR(0)	AR(1)	AR(2)	AR(3)	AR(4)
MLL Chib (MH)	-279,12	-269,7	-269,71	-273,72	-275,46
MLL Chib (Gibbs)	-279,12	-269,7	-269,71	-273,72	-275,46

Same Estimates from Gibbs or MH samplers

Marginal likelihood depends on the prior!

- Test the order of the AR process with different priors

GARCH estimation by MH MCMC

- **Change the directory and go to *GARCH_with_MH***
- **Import financial data by clicking on *SP500_percentage_returns.mat***
- **Main matlab program *MCMC_GARCH_RW***

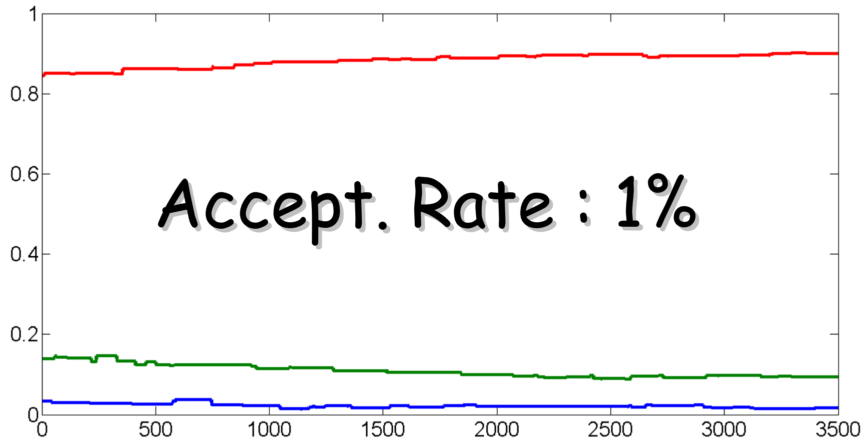
  Estimate a GARCH(1,1) model with
    - non adaptive RW
    - adaptive RW

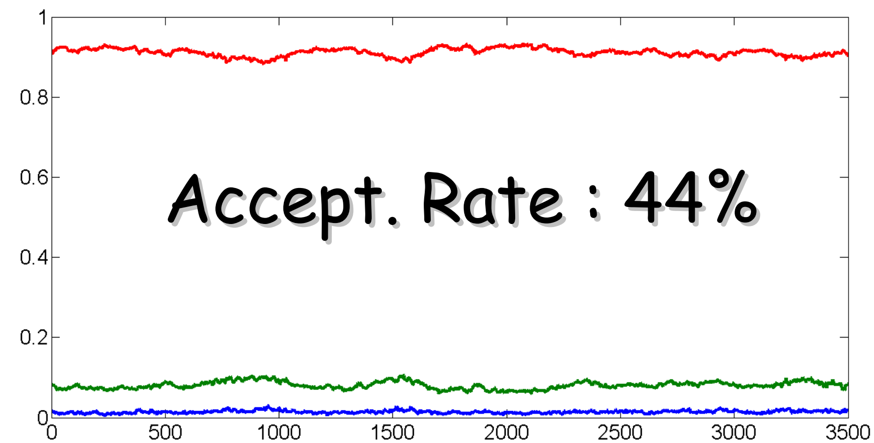
Inputs :

- Y : financial time series **(here SP500)**
- nb_MCMC : number of MCMC iterations
- RW_step : Variance of the proposal distribution
- Graph : Convergence graphics

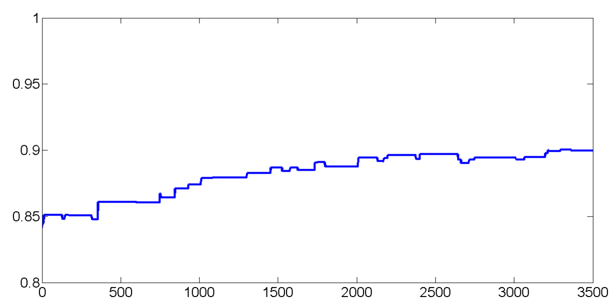
GARCH estimation by MH MCMC

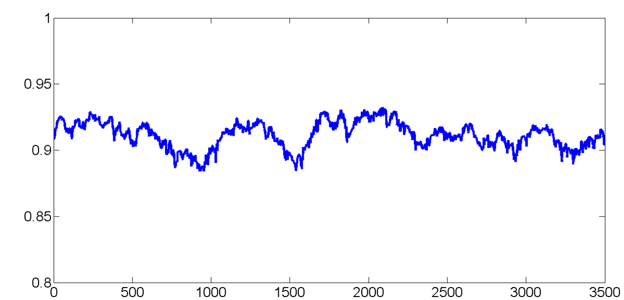
- Choose a RW variance and run the program
- [Simu] = MCMC_GARCH_RW(SP500,5000,0.1)





- Focus on parameter β in $\sigma_t^2 = \omega + \alpha y_{t-1}^2 + \beta \sigma_{t-1}^2$





Change-Point AR models

CP-AR models with CGS

- **Change the directory and go to *CP_models_with_CGS***

- **Import data by clicking on US_GDP_percentage.*mat***

- **Main matlab program**
  Gibbs_regression_Carlin_Gelman_Smith
  - Estimates a CP model with 2 regimes using CGS's algorithm

Inputs :

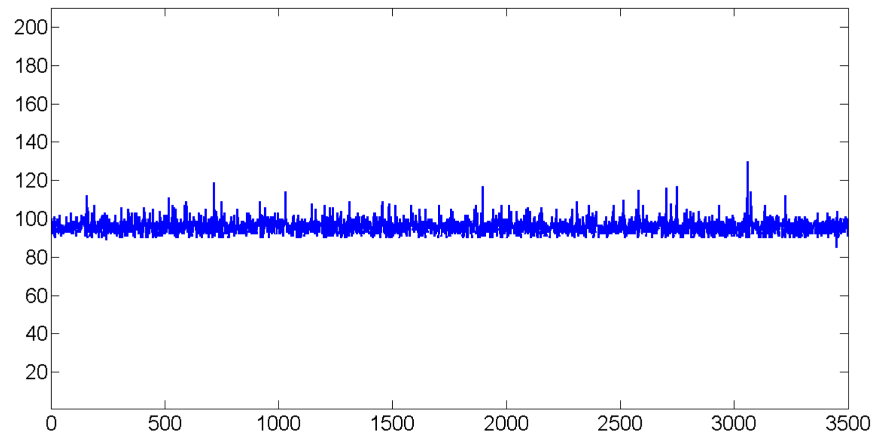
- Y : a time series **(here US_GDP_growth)**
- X : explanatory variables
- nb_MCMC : number of MCMC iterations

- **Program for estimating a CP-AR(q) model**
  Launch_CP_AR_Carlin_Gelman_Smith

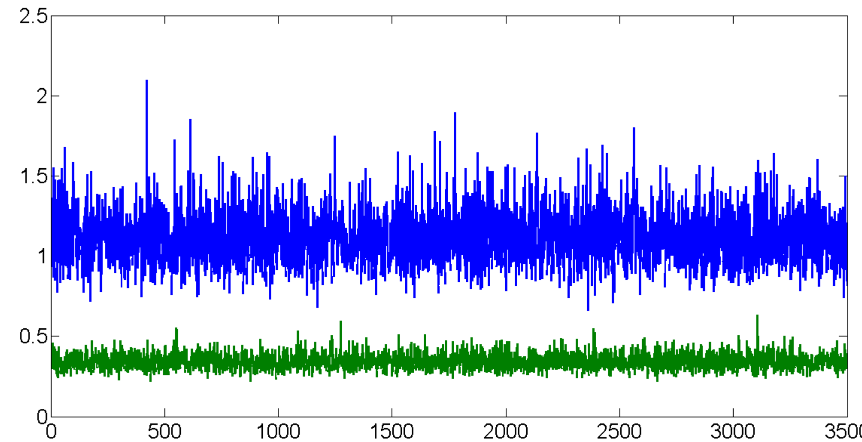
CP-AR models with CGS

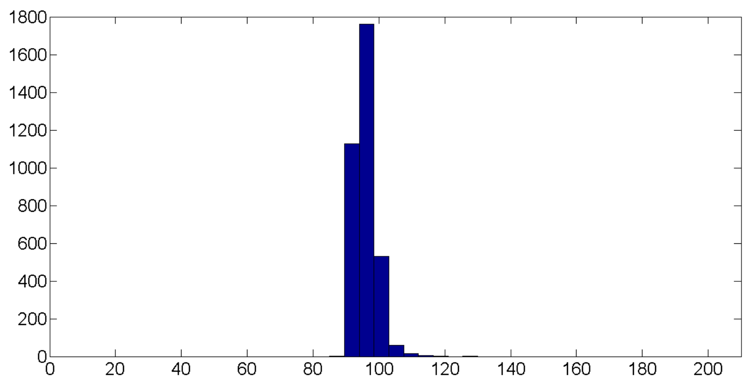
- Run an estimation of a CP-AR(1) model
  [Simu] = launch_CP_AR_Carlin_Gelman_Smith(US_GDP_growth,1,5000)

plot(Simu.post_tau')



plot(Simu.post_sigma')





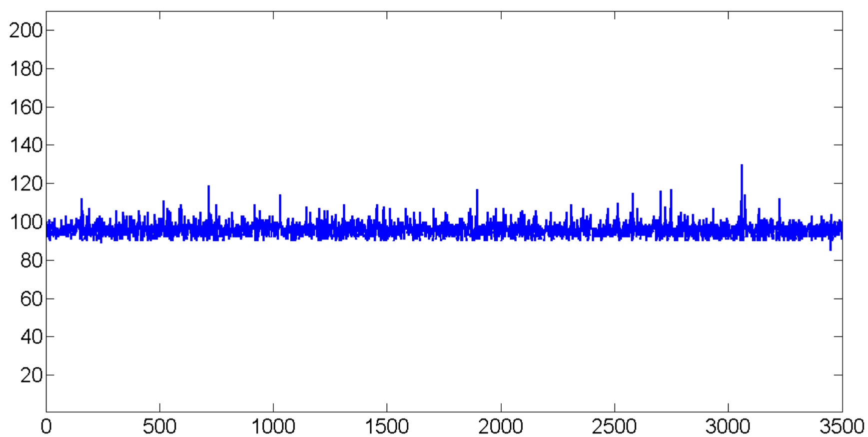
- Great moderation
- Not a symmetric distribution

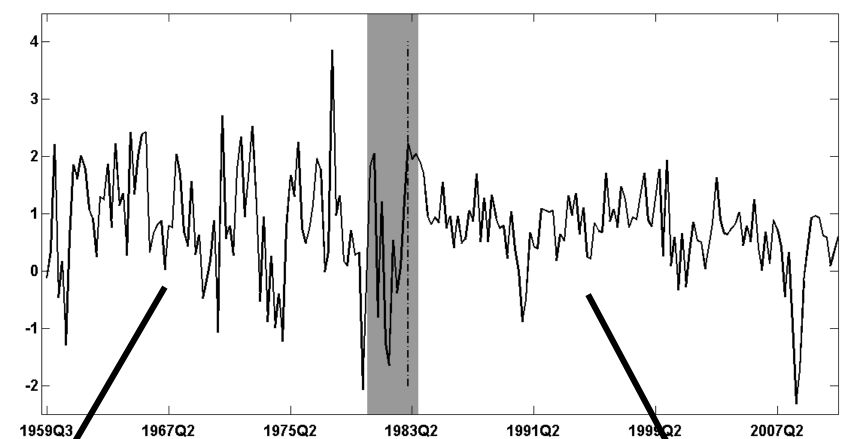
CP-AR models with CGS

- Run an estimation of a CP-AR(1) model

  [Simu] =
  launch_CP_AR_Carlin_Gelman_Smith(US_GDP_growth,1,5000)

plot(Simu.post_tau')





$E(\sigma_1^2|Y_{1:T}) \approx 1.11$ $E(\sigma_2^2|Y_{1:T}) \approx 0.34$

CP-AR models with Chib

- **Change the directory and go to *CP_models_with_Chib***
- **Import data by clicking on US_GDP_percentage.*mat***
- **Main matlab program *Gibbs_regression_chib***
  - Estimates a CP model with k regimes using Chib's algorithm

Inputs :

- Y : a time series **(here US_GDP_growth)**
- X : explanatory variables
- Regime : number of regimes
- nb_MCMC : number of MCMC iterations
- MLL_computation : =1 if MLL must be estimated

CP-AR models with Chib

- **Main matlab program** ***Gibbs_regression_chib***
  - Estimates a CP model with k regimes using Chib's algorithm
- **Matlab program** ***launch_CP_model_estimations***
  - Estimates CP-AR models from one up to k regimes

Inputs :

- Y : a time series **(here US_GDP_growth)**
- AR_lags : The order of the AR process
- upper_bound_regime : Max. considered number of regimes
- nb_MCMC : number of MCMC iterations

CP-AR models with Chib

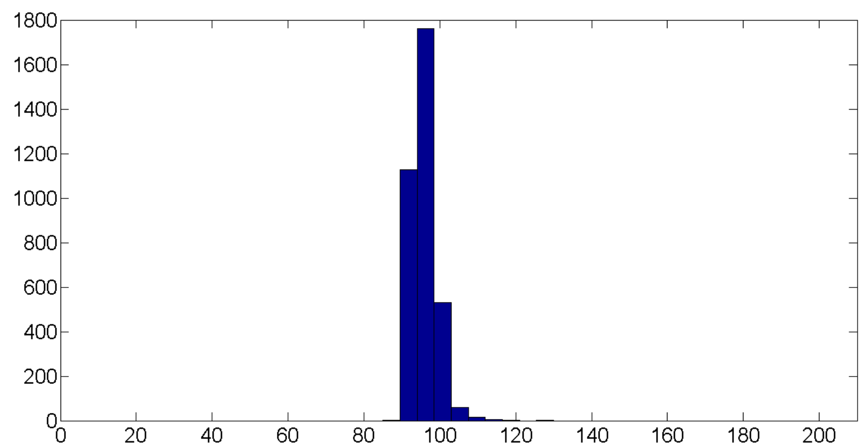
- Run an estimation of CP-AR(1) models from one to 3 regimes

  [Simu MLL] = launch_CP_model_estimations(US_GDP_growth,1,3,10000)

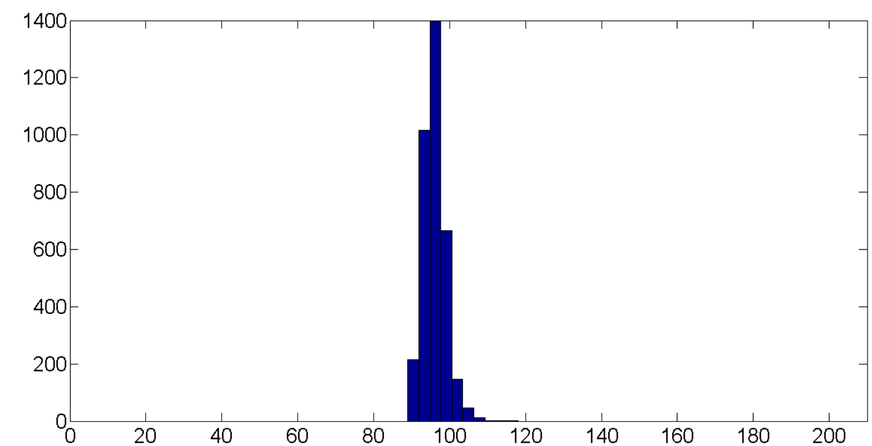
  nb_MCMC

- Two regimes

Griddy-Gibbs



Chib



Gibbs sampler

- **The model**

$$\begin{cases} y_t &= \beta'_{s_t} x_t + \epsilon_t \\ \epsilon_t &\sim \text{i.i.d.} N(0, \sigma^2_{s_t}) \end{cases}$$

- **The prior distributions**

$\beta_i \sim N(\beta_0, \Sigma_0) \ \ \forall\ i \in [1, K+1]$ **and** $\sigma_i^2 \sim IG(IG_a, IG_b) \forall i \in [1, K+1]$

Transition prob. From *i* to *i* : $p_{ii} \sim \text{Beta}(\alpha_p, \beta_p)$

Trans. State :

$$\begin{cases} P(s_t = s_{t-1} | s_{t-1}, P) = p_{s_{t-1}, s_{t-1}} \\ P(s_t = s_{t-1} + 1 | s_{t-1}, P) = 1 - p_{s_{t-1}, s_{t-1}} \forall s_{t-1} \in [1, K] \end{cases}$$

Gibbs sampler

- **The model**

$$\begin{cases} y_t &= \beta'_{s_t} x_t + \epsilon_t \\ \epsilon_t &\sim \text{i.i.d.} N(0, \sigma^2_{s_t}) \end{cases}$$

Priors in the program : Gibbs_regression_chib

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%% Set the Hyper-parameters
%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%% Prior :
%%% beta_i ~ N(beta_0,var_uninformative)
%%% sigma^-2 ~ G(a,b)
%%% p_ii ~ Beta(alpha,beta)
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
var_uninformative = 10; %% We fix the variance of each beta equal to var_u
beta_0 = zeros(dimension,1);
inv_Sigma_0 = diag(ones(dimension,1)*(1/var_uninformative));
det_inv_Sigma_0 = det(inv_Sigma_0);

%%% hyper-parameters for the variances
IG_b = 1;
IG_a = 1;

%%% hyper-parameters for each probability of staying in the same regime
alpha_p = 10;
beta_p = 1;
```

$$\beta_i \sim N(\beta_0, \Sigma_0)$$

$$\sigma_i^2 \sim IG(IG_a, IG_b)$$

$$p_{ii} \sim \text{Beta}(\alpha_p, \beta_p)$$

Forward-Backward

- **Gibbs step :** $S_{1:T}|Y_{1:T}, \beta_1, ..., \beta_{K+1}, \sigma_1^2, ..., \sigma_{K+1}^2, P$

Program for sampling a state vector : Forward_Backward

Two steps :

1) Compute the forward prob. $f(s_t|Y_{1:t}) \forall t \in [1, T]$

2) Sample a state using the decomposition

$$\pi(S_{1:T}|Y_{1:T}) = \pi(s_T|Y_{1:T})\pi(s_{T-1}|Y_{1:T}, s_T)...\pi(s_1|Y_{1:T}, S_{2:T})$$

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%% Backward algorithm : sampling a state vector
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
sn = zeros(T,1);
sn(T) = regime;
for t=T-1:-1:1
   etat_fut = sn(t+1);
   back = forward(t,:)'.*P(:,etat_fut);
   back = back/sum(back);
   sn(t) =  multinomialrnd(back);
end
```

$$\pi(s_t|Y_{1:T}, S_{t+1:T}) \propto f(s_t|Y_{1:T})f(s_{t+1}|s_t)$$

Marginal likelihood

- **Local formula :**

Likelihood (by F-B) — Priors

$$f(Y_{1:T}) = \frac{f(Y_{1:T}|\beta_1^*, ..., \beta_{K+1}^*, \sigma_1^{2,*}, ..., \sigma_{K+1}^{2,*}, P^*) f(\beta_1^*, ..., \beta_{K+1}^*, \sigma_1^{2,*}, ..., \sigma_{K+1}^{2,*}, P^*)}{\pi(\beta_1^*, ..., \beta_{K+1}^*, \sigma_1^{2,*}, ..., \sigma_{K+1}^{2,*}, P^*|Y_{1:T})}$$

- **Third term :**

$$\pi(\beta_1^*, ..., \beta_{K+1}^*, \sigma_1^{2,*}, ..., \sigma_{K+1}^{2,*}, P^*|Y_{1:T}) = \underbrace{\pi(P^*|Y_{1:T})}_{1} \underbrace{\pi(\sigma_1^{2,*}, ..., \sigma_{K+1}^{2,*}|P^*, Y_{1:T})}_{2} \underbrace{\pi(\beta_1^*, ..., \beta_{K+1}^*|\sigma_1^{2,*}, ..., \sigma_{K+1}^{2,*}, P^*, Y_{1:T})}_{3}$$

1) $$\pi(P^*|Y_{1:T}) = \int \pi(P^*|Y_{1:T}, S_{1:T})\pi(S_{1:T}|Y_{1:T})dS_{1:T}$$

$$\pi(P^*|Y_{1:T}) \approx \frac{1}{N}\sum_{i=1}^{N} \pi(P^*|Y_{1:T}, S_{1:T}^i)$$

Marginal likelihood

2) $$\pi(\sigma_1^{2,*},...,\sigma_{K+1}^{2,*}|P^*,Y_{1:T}) = \int \pi(\sigma_1^{2,*},...,\sigma_{K+1}^{2,*}|P^*,\beta_1,...,\beta_{K+1},S_{1:T},Y_{1:T}) \pi(S_{1:T},\beta_1,...,\beta_{K+1}|P^*,Y_{1:T})dS_{1:T}d\beta_1...d\beta_{K+1}$$
$$\approx \frac{1}{G_1}\sum_{i=1}^{G_1}\pi(\sigma_1^{2,*},...,\sigma_{K+1}^{2,*}|P^*,\beta_1^i,...,\beta_{K+1}^i,S_{1:T}^i,Y_{1:T})$$

- **Run an auxiliary MCMC with fixed P***

3) $$\pi(\beta_1^*,...,\beta_{K+1}^*|P^*,\sigma_1^{2,*},...,\sigma_{K+1}^{2,*},Y_{1:T}) = \int \pi(\beta_1^*,...,\beta_{K+1}^*|S_{1:T},\sigma_1^{2,*},...,\sigma_{K+1}^{2,*},Y_{1:T}) \pi(S_{1:T}|P^*,\sigma_1^{2,*},...,\sigma_{K+1}^{2,*},Y_{1:T})dS_{1:T}$$
$$\approx \frac{1}{G_2}\sum_{i=1}^{G_2}\pi(\beta_1^*,...,\beta_{K+1}^*|S_{1:T}^i,\sigma_1^{2,*},...,\sigma_{K+1}^{2,*},Y_{1:T})$$

- **Run second auxiliary MCMC with fixed** $\mathrm{P}^*,\sigma_1^{2,*},...,\sigma_{K+1}^{2,*}$

Model selection

- Run an estimation of CP-AR(1) models from one to 3 regimes
  [Simu MLL] = launch_CP_model_estimations(US_GDP_growth,1,3,10000)

Uninformative prior :

$\beta_i \sim N(\underline{0}, 1000 I_d) \qquad \sigma_i^{-2} \sim G(0.01, 100) \qquad p_{ii} \sim \mathrm{Beta}(0.5, 0.5)$

#Regime	1	2	3
MLL	-275.57	-269.25	-275.10

Informative prior :

$\beta_i \sim N(\underline{0}, I_d) \qquad \sigma_i^{-2} \sim G(1, 1) \qquad p_{ii} \sim \mathrm{Beta}(0.5, 0.5)$

#Regime	1	2	3
MLL	-265.28	-249.96	-243.52

Model selection

- Choose your prior according to 'your break sensitivity'

- Or use another criterion such as the predictive likelihood

→ Less impacted by the prior distributions

$$
\begin{aligned}
f(Y_{t+1:T}|Y_{1:t}) &= \frac{f(Y_{1:T})}{f(Y_{1:t})} \\
&= \frac{f(Y_{1:T}|\theta^*)f(\theta^*)}{\pi(\theta^*|Y_{1:T})}\frac{\pi(\theta^*|Y_{1:t})}{f(Y_{1:t}|\theta^*)f(\theta^*)} \\
&= \frac{f(Y_{1:T}|\theta^*)}{f(Y_{1:t}|\theta^*)}\frac{\pi(\theta^*|Y_{1:t})}{\pi(\theta^*|Y_{1:T})}
\end{aligned}
$$

No prior distributions!
Impact through the posterior